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ON LINEAR COMBINATORICS III. FEW DIRECTIONS AND DISTORTED LATTICES

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We study finite configurations of the Euclidean plane which determine few (i.e. a linear number of) directions. A structure theorem is proven on those such sets which have many collinear points.

Our main tools will be certain "distorted" lattices which contain many rich lines (i.e. straight lines incident upon many points of the lattice).

1. Introduction

This article concludes the sequence of papers we started with parts I and II [2, 3].

We are going to study finite subsets of the Euclidean plane which only determine a linear number of directions; especially those which, moreover, have many collinear points. A characterization in terms of generalized arithmetic and geometric progressions will be shown. (All notions are introduced in Section 2.)

The tools we use (including the Main Lemma) concern some natural generalizations of square lattices (called "distorted lattices") which have many "rich" diagonals (i.e. lines with many points of these configurations).

Our results are structure theorems specific to Euclidean geometry — they do not hold true within finite planes. (See the introduction of Part I for a brief historic overview of certain such results.)

1.1. Sets with few directions

Definition 1. For a finite point set $\mathcal{A}\subset\mathbb{R}^k$, we put

$$D(\mathcal{A}) \stackrel{\text{def}}{=} \#\{\text{directions of segments } \overline{A_1A_2} \mid A_1, A_2 \in \mathcal{A},\ A_1 \neq A_2\}.$$

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For the sake of simplicity, we do not distinguish segments $\overline{A_1A_2}$ and $\overline{A_2A_1}$; thus two segments have equal directions iff they are parallel.

The study of sets which determine few distinct directions was initiated by Scott [7]. He conjectured that $D(\mathcal{A}) \geq |\mathcal{A}| - 1$ for any non-collinear planar point set. This was settled in the affirmative by Ungar [9], see also [1]. Sets for which equality holds are called *critical* by Jamison [5] and those with one more directions, i.e. $D(\mathcal{A}) = |\mathcal{A}|$, are *near-critical*. He gives an overview of the known critical and near-critical configurations of the Euclidean plane and, among others, characterizes those such configurations $\mathcal{A}$ which lie on the union of two or three straight lines. His two basic structures are:

(a) copies of an arithmetic progression on each of two or three parallel lines (with the starting points fitted appropriately) — Jamison calls them "bicolumnar" and "tricolumnar" arrays, respectively; and

(b) copies of a geometric progression on each of the four half-lines of the two coordinate axes plus the origin — an "exponential cross".

The above structures will be referred to as "Jamison configurations".

It is natural to ask questions like

"what is the structure of $\mathcal{A}$ if it determines slightly (or much but not too much) more slopes than critical or near-critical ones?"

However, little is known about plane sets with $D(\mathcal{A}) = |\mathcal{A}| + 1$ or $|\mathcal{A}| + 100$, let alone $D(\mathcal{A}) = 2|\mathcal{A}|$ or $1000|\mathcal{A}|$.

In this paper we show that (roughly speaking), beyond the above Jamison configurations, no essentially different structures can satisfy even the much weaker requirement $D(\mathcal{A}) \leq C|\mathcal{A}|$, provided that a good proportion of $\mathcal{A}$ is collinear. To this end, we generalize these configurations in two ways:

(i) we allow several (but a bounded number of) parallel or concurrent lines (see Figure 1);

(ii) we place certain "generalized" arithmetic or geometric progressions on them (see next section).

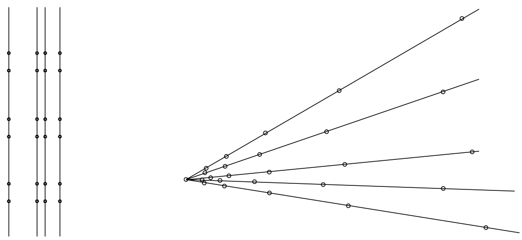

Fig. 1. Generalized Jamison configurations

Theorem 1. *Let $C > 1$ be fixed; $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathbb{R}^2$ with $n \leq |\mathcal{A}_1|, |\mathcal{A}_2| \leq Cn$ and let $\mathcal{A}_1$ be collinear.*

If $D(\mathcal{A}) \leq Cn$ then $\mathcal{A}$ is contained in a "generalized Jamison configuration".

(See the next section for the definitions and Theorem 2 for a slightly more general statement.)

2. Definitions

2.1. Arithmetic and geometric GPs

Following [8] and [6], we define generalized arithmetic and geometric progressions.

Definition 2. Let d and $n_1, n_2, \ldots, n_d$ be positive integers and $\Delta_1, \Delta_2, \ldots, \Delta_d$ arbitrary real numbers. We shall call the set

$$\mathcal{G} = \left\{ \sum_{i=1}^{d} k_i \cdot \Delta_i \ ; \ 0 \leq k_i < n_i \text{ for } i = 1 \ldots d \right\}$$

a *generalized arithmetic progression* of dimension d and parameters n_i and Δ_i. Similarly — instead of differences Δ_i — with quotients $q_1, q_2, \ldots, q_d$ positive reals, $\mathcal{G}$ is a *generalized geometric progression* if

$$\mathcal{G} = \{ \pm \prod_{i=1}^{d} q_i^{k_i} \ ; \ 0 \leq k_i < n_i \text{ for } i = 1 \ldots d \}.$$

The $\pm$ sign is placed in the definition for convenience; we just wanted to avoid having to play around with quotients of different signs.

We shall use the shorthands "arithmetic GP" and "geometric GP" for the above structures.

Remark 3. Note that the sums or products in the above definition may not be distinct, i.e. $|\mathcal{G}| < \prod n_i$ or $|\mathcal{G}| < 2\prod n_i$ are possible, respectively.
In what follows, $\mathcal{G}_{d,n}$ will denote an arithmetic or geometric GP of dimension *not exceeding* d and size *at most* n. For short, we shall also use expressions like "there exists an arithmetic or geometric $\mathcal{G}_{d,n}$".

The following statement is, probably, folklore. (It is really obvious if the sums or products in Definition 2 are all distinct. Otherwise, see e.g. [3], Proposition 3 for a simple proof.)

Proposition 4. *If $\mathcal{G}$ is an arithmetic or geometric GP of dimension d then $|\mathcal{G} \pm \mathcal{G}| \leq 2^d |\mathcal{G}|$ or $|\mathcal{G} \cdot \mathcal{G}|, |\mathcal{G}/\mathcal{G}| \leq 2^d |\mathcal{G}|$, respectively.* ∎

(Here the algebraic operations "$\pm$", "$\cdot$" and "/" are meant element-wise, i.e. they must be performed for all pairs $(g_1, g_2) \in \mathcal{G}$.)

Based upon the above notions we introduce generalized Jamison configurations.

2.2. Generalized Jamison configurations

Definition 5. Let C be a positive integer. $\mathcal{J} \subset \mathbb{R}^2$ is an *arithmetic GP-type generalized Jamison configuration* of thickness C, based upon an arithmetic GP $\mathcal{G}$, if there exist vectors $\overline{v}, \overline{h}_1, \ldots, \overline{h}_C \in \mathbb{R}^2$ such that

$$\mathcal{J} = \bigcup_{i=1}^{C} (\mathcal{G}\overline{v} + \overline{h}_i).$$

(In other words, $\mathcal{J}$ consists of C copies of an arithmetic GP, one on each of C parallel lines.)

Definition 6. Let C be a positive integer. $\mathcal{J}$ is a *geometric GP-type generalized Jamison configuration* of thickness C and center $\overline{b} \in \mathbb{R}^2$, based upon a geometric GP $\mathcal{G}$, if there exist vectors $\overline{a}_1, \ldots, \overline{a}_C \in \mathbb{R}^2$ such that

$$\mathcal{J} = \{\overline{b}\} \cup \bigcup_{i=1}^{C} \{g\overline{a}_i + \overline{b}\ ;\ g \in \mathcal{G}\}.$$

(Here we have copies of a geometric GP on each of C concurrent lines, together with the point of intersection, which we call the "center".)

Remark 7. It follows from Proposition 4, that generalized Jamison configurations cannot determine more than $\binom{C}{2} 2^d |\mathcal{G}| + C$ directions.

3. Results and further problems

Throughout this paper we use the convention of denoting by c, C or d those constants which are independent of n (the asymptotic size of our structures). Also, we use c^*, C^* or d^* for those which may depend on c, C and/or d but not on n.

Moreover, in what follows, we shall assume that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathbb{R}^2$ with $n \leq |\mathcal{A}_1|, |\mathcal{A}_2| \leq Cn$ and $\mathcal{A}_1 \subset l$ for some straight line l while $\mathcal{A}_2 \cap l = \emptyset$. It will turn out that for such sets $\mathcal{A}$, even the weaker assumption

$$D(\mathcal{A}_1, \mathcal{A}_2) \stackrel{\text{def}}{=} \#\{\text{directions of segments } \overline{A_1 A_2} \mid A_i \in \mathcal{A}_i\} \leq Cn$$

is strong enough for a complete characterization of structures which satisfy it.

Theorem 2. *If $D(\mathcal{A}_1, \mathcal{A}_2) \leq Cn$ then $\mathcal{A}$ is contained in a generalized Jamison configuration of thickness C^* based upon an arithmetic or geometric $\mathcal{G}_{d^*, C^{**}n}$.*

(Note that a positive proportion of a generalized Jamison configuration really determines a linear number of distinct directions by Proposition 4.)

The question, whether or not $\mathcal{A}$ must be located on C^* lines, was originally asked by P. Hajnal.

Also, a collinear *sub*structure will be found if we just assume

$$D_E(\mathcal{A}_1,\mathcal{A}_2) \stackrel{\text{def}}{=} \#\{\text{directions of segments } \overline{A_1A_2} \mid (A_1,A_2)\in E\} \le Cn,$$

for some (not too small) $E\subset\mathcal{A}_1\times\mathcal{A}_2$.

Theorem 3. *If $D_E(\mathcal{A}_1,\mathcal{A}_2)\le Cn$ for some $E\subset\mathcal{A}_1\times\mathcal{A}_2$ with $|E|\ge n^2$ then there is a collinear subset $\mathcal{A}_2^*\subset\mathcal{A}_2$ such that $|E\cap(\mathcal{A}_1\times\mathcal{A}_2^*)|\ge c^*n^2$.*
(Thus, of course, $\mathcal{A}_2^*$ contains $(c^*/C)n=c^{**}n$ or more points.)

Even the following may be true.

Conjecture 1. *Let $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ and E be as above. Assume, moreover, that all $a_1\in\mathcal{A}_1$ as well as all $a_2\in\mathcal{A}_2$ occur in not less than n pairs $(a_1,a_2)\in E$. If $D_E(\mathcal{A}_1,\mathcal{A}_2)\le Cn$, then $\mathcal{A}$ can be covered with $C^*=C^*(C)$ straight lines.*

We pose yet another problem on sets with few directions.

Conjecture 2. *For every $C>0$ there is an $n_0=n_0(C)$ with the following property. If $\mathcal{A}\subset\mathbb{R}^2$ with $|\mathcal{A}|\ge n_0$ and $D(\mathcal{A})\le C|\mathcal{A}|$ then $\mathcal{A}$ contains six points of a (possibly degenerate) conic.*
(As usual, a pair of lines is considered a degenerate conic.)

Of course, there exist several conconic examples with few directions. E.g. a set of n equally spaced points on a circle (or the affine image thereof) determines at most $2n$ directions. Similarly, so do point sets on the parabola $y=x^2$ with x-coordinates from an arithmetic progression. Finally, yet another such example consists of points on (say, the positive arc of) the hyperbola $y=1/x$, with x-coordinates from a geometric progression.

It is very likely that Conjecture 2 holds true for any number in place of six, for $|\mathcal{A}|$ large enough. (It was pointed out to us by M. Simonovits, that one cannot expect $c^*|\mathcal{A}|$ conconic points in general, as shown by a $k\times k$ lattice.) However, some $|\mathcal{A}|^\alpha$ such points may exist, for a suitable $\alpha=\alpha(C)>0$. Perhaps even $c^*|\mathcal{A}|$ can be found, provided that $\mathcal{A}$ is the vertex set of a convex polygon.

A weaker version could be the following.

Conjecture 3. *If a non-collinear set $\mathcal{A}$ of n points is located on an irreducible algebraic curve of degree r, and $D(\mathcal{A})\le Cn$, then the curve must be a conic, provided that $n>n_0(r,C)$.*

In order to support Conjectures 2 and 3 we show that, of all polynomial curves $y=p(x)$ (where $p\in\mathbb{R}[x]$), only parabolae can accommodate n non-collinear points with at most Cn directions, provided that n is large enough as compared to the degree of p.

Theorem 4. *Let $r \geq 3$ be an integer and $C > 1$. Then there exists an $n_0 = n_0(r, C)$ with the property that no polynomial curve of degree r (defined by an equation $y = p(x)$) can contain a set of $n > n_0$ points $\mathcal{A} = \{A_1, \ldots, A_n\}$ with $D(\mathcal{A}) \leq Cn$.*

The proof of the Theorems can be found in Section 5.

4. Distorted lattices and the Lattice Lemma

In this section we introduce some lattice-like structures and our main tool, the Lattice Lemma.

4.1. Distorted lattices and rich lines

Definition 8. For arbitrary $X, Y \subset \mathbb{R}$, we call the point set $\mathcal{D} = X \times Y \subset \mathbb{R}^2$ a *distorted lattice.*

Definition 9. Let t be an arbitrary natural number, $\mathcal{D}$ a distorted lattice, and l a straight line which is neither vertical nor horizontal. Then l is called t-rich in $\mathcal{D}$, if $|l \cap \mathcal{D}| \geq t$.

Beyond the usual square (or rectangle) lattices, other interesting species of "regular" distorted lattices are what we call "geometric lattices": $X \times X$ for some quotient $q > 0$ and $X = \{0\} \cup \{q^i \mid i = 1 \ldots n-1\}$ (i.e. a geometric progression, with 0 added; see Figure 2.c).

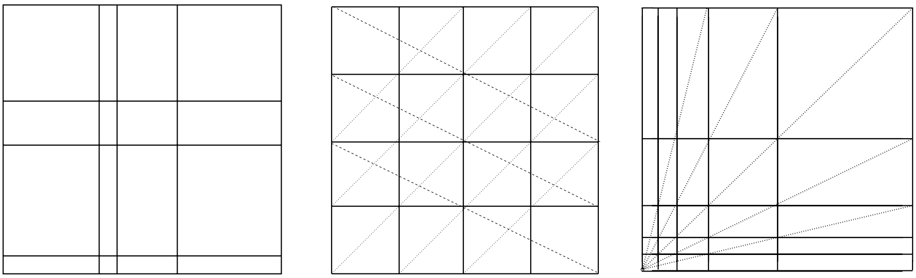

Fig. 2. Distorted lattices with some 3-rich lines marked.
a) A general distorted lattice **b)** A square lattice **c)** A geometric lattice with $q = 2$

The dotted rich lines in the figure might be called "blow-up lines" since, while you blow the lattice up by, say, a factor of q from the origin, the lattice points move along these lines.

Our main lemma describes the structure of distorted lattices of size $n \times n$ or less which contain many cn rich lines, (for $c > 0$ fixed) — or, using a notation more convenient for further applications, $Cn \times Cn$ lattices with many n-rich lines — in terms of arithmetic or geometric GPs, and their bounded unions: semiregular lattices.

4.2. Semiregular lattices

It is easy to check that the number of n-rich lines is at least c^*n in $Cn\times Cn$ distorted lattices of type $\mathcal{G}\times\mathcal{G}$ for $\mathcal{G}$ an arithmetic or geometric GP of fixed dimension.

This observation holds true for the union of a bounded number of certain such lattices, as well. We shall call these structures "semiregular lattices".

Definition 10. Let C be a positive integer. $\mathcal{D}$ is a *semiregular arithmetic GP-type lattice* with parameter C, based upon an arithmetic GP $\mathcal{G}$, if there exist real numbers $s_1,\ldots,s_C$ such that

$$\mathcal{D}=\mathcal{G}\times\bigcup_{i=1}^{C} s_i\mathcal{G}.$$

Similarly, $\mathcal{D}$ is a *semiregular geometric GP-type lattice* with parameter C, based upon a geometric GP $\mathcal{G}$, if there exist real numbers $u,v_1,\ldots,v_C$ such that

$$\mathcal{D}=\Big((\mathcal{G}\cup\{0\})+u\Big)\times\bigcup_{i=1}^{C}\Big((\mathcal{G}\cup\{0\})+v_i\Big).$$

Remark 11. Adding C numbers to an arithmetic GP or multiplying a geometric GP by C numbers makes no structural change; the results will be contained in a similar progression of dimension C higher and size at most 2^C times the original. That is why we only need to multiply arithmetic GPs and add to geometric GPs.

4.3. The Lattice Lemma

Now we state our main lemma. Of its three parts, (i) and (ii) are short and simple while the third one is rather technical. (If you think of the $|X|$-rich lines mentioned there as graphs of linear functions on the domain X then Y^* is simply the union of their ranges.)

Lemma 12. **(Lattice Lemma)** *For every $C>0$ there are $C'=C'(C)$, $c^*=c^*(C)$, $C^*=C^*(C)$ and $d^*=d^*(C)$ with the following properties.*
Let $\mathcal{D}=X\times Y$ be a distorted lattice with $n\le|X|,|Y|\le Cn$. Then

(i) *$\#\{n$-rich lines of $\mathcal{D}\}\le C'n$;*
(ii) *if $\#\{n$-rich lines of $\mathcal{D}\}\ge n$ then at least c^*n of these lines are concurrent or parallel;*
(iii) *if $\#\{|X|$-rich lines of $\mathcal{D}\}\ge n$ then there exists a $Y^*\subset Y$, such that*

- *the original $|X|$-rich lines are still $|X|$-rich in $\mathcal{D}^*\stackrel{\text{def}}{=}X\times Y^*$;*

- *$\mathcal{D}^*$ is contained in a semiregular lattice with parameter at most C^*, based upon an arithmetic or geometric $\mathcal{G}^*_{d^*,C^*n}$.*

Also, the equations of all the $|X|$-rich lines must be of the form

$$x \mapsto s_i(x+g) \text{ — in case of arithmetic GPs; or}$$

$$x \mapsto g(x-u)+v_i \text{ — in case of geometric GPs,}$$

where $g \in \mathcal{G}^*$ and the s_i or u and v_i are as in Definition 10;

Proof. Statement (i) is just Lemma 20 of Part I [2] for $M = Cn$ and $c = 1/C$; Statement (ii) is equivalent to Theorem 2 of Part I [2].
Statement (iii) follows from Theorem 3 of Part II [3] by letting $\mathcal{G}^* = \mathcal{G}+\mathcal{G}$ (resp. $\mathcal{G}^* = \mathcal{G}\cdot\mathcal{G}$) and using the fact that $\mathcal{G}^*$, again, is an arithmetic GP (resp. geometric GP), provided that $\mathcal{G}$ is one. ∎

It would be nice to know whether, under the assumptions of part (ii) of Lemma 12, the n-rich lines must come from C^{**} bundles, each parallel or concurrent. (Both types may occur in one structure.) This statement would be almost as strong as that of part (iii), from a much weaker assumption.

Conjecture 4. *Let $\mathcal{D} = X \times Y$ be a distorted lattice with $n \le |X|, |Y| \le Cn$. If $\#\{n\text{-rich lines of } \mathcal{D}\} \ge n$ then there exist $X^* \subset X$ and $Y^* \subset Y$, such that*

- *the original n-rich lines are still n-rich in $\mathcal{D}^* \stackrel{\text{def}}{=} X^* \times Y^*$;*
- *$\mathcal{D}^*$ is contained in the union of C^{**} semiregular lattices with parameters at most C^* each, based upon arithmetic and/or geometric GPs of dimensions not exceeding d^* and sizes at most C^*n.*

5. Proof of the Theorems

5.1. Proof of Theorems 2 and 3

Both Theorems will follow from the Lattice Lemma. (Moreover, Theorem 2 immediately implies Theorem 1, as well.)

First, without loss of generality we may assume that the line l which contains $\mathcal{A}_1$ is the x-axis (otherwise we shift or rotate $\mathcal{A}$).

Throughout the rest of the proof we shall work in the projective plane.

1. Apply a polarity
$$(a,b,c) \;\leftrightarrow\; cx+by+az=0,$$
where the point with projective coordinates (a,b,c) will correspond to the line on the right and vice versa. (This mapping is known to be incidence preserving.)

2. The following are obvious.
   - For points of $\mathcal{A}_1$,
   $$(a,0,1) \;\leftrightarrow\; x+0y+az=0,$$
   i.e. in Cartesian coordinates, $(a,0) \leftrightarrow x=-a$.
   - The point which corresponds to the line at infinity is
   $$(1,0,0) \;\leftrightarrow\; z=0.$$
   - A segment (or, rather, a line which connects two points), corresponds to the point of intersection of those lines which correspond to the points.
   - The slope s (i.e. the point on the line at infinity which represents this slope) corresponds to
   $$(1,s,0) \;\leftrightarrow\; 0x+sy+z=0,$$
   i.e. in Cartesian coordinates, (slope s) $\leftrightarrow$ $y=-1/s$, which is a horizontal line.
   - The points of $\mathcal{A}_2$ are neither on the x-axis, nor on the line at infinity, therefore they correspond to lines which are neither vertical nor horizontal.
3. To prove Theorem 3, assume without loss of generality that each $A_2 \in \mathcal{A}_2$ occurs in n or more pairs $(A_1,A_2)\in E$. (Otherwise let $\hat{n}=n/(2C)$ and keep on deleting those $A_2\in\mathcal{A}_2$ which occur in less than this many pairs. You cannot delete everything.) Then use part (ii) of the (Lattice) Lemma 12 for the polar structure with the new values of $\hat{n}$ above and $\hat{C}=2C^2$.

A parallel bundle has to come from points of $\mathcal{A}_2$ on a horizontal line; a concurrent bunch from a non-horizontal line. This proves Theorem 3.

4. To prove Theorem 2, first observe that the assumption $D(\mathcal{A}_1,\mathcal{A}_2) \le Cn$ is equivalent to the fact that part (iii) of the (Lattice) Lemma 12 can be applied to the polar structure. Thus it must be contained in a semiregular lattice based upon a $\mathcal{G}^*$.
   (a) If $\mathcal{G}^*$ is an arithmetic GP then the lines which correspond to $\mathcal{A}_2$ are of the form
   $$x \mapsto s_i(x+\hat{g}) \qquad (\hat{g}\in\hat{\mathcal{G}}).$$
   Let $t_i=-1/s_i$ and $\overline{h_i}=(0,t_i)$. Then, for the above lines,
   $$x+t_iy+\hat{g}z=0 \;\leftrightarrow\; (\hat{g},t_i,1)\in\mathcal{G}^*+\overline{h_i}.$$
   (b) If $\mathcal{G}^*$ is a geometric GP then the lines which correspond to $\mathcal{A}_2$ are of the form
   $$x \mapsto \hat{g}(x-u)+v_i \qquad (\hat{g}\in\mathcal{G}^*).$$
   Let $\mathcal{G}=\mathcal{G}^*/\mathcal{G}^*$, $\overline{a}_0=(-1,0)$, $\overline{a}_i=(v_i,-1)$ for $1\le i\le\hat{C}$, and $\overline{b}=(-u,0)$. Then, for the above lines and $1/g=\hat{g}\in\mathcal{G}^*$,
   $$x-gy+(v_ig-u)z=0 \;\leftrightarrow\; (v_ig-u,-g,1)\in\mathcal{G}\overline{a}_i+\overline{b}.$$

Moreover, the points of $\mathcal{A}_1$ correspond to vertical lines $x = \hat{g} + u$ ($\hat{g} \in \mathcal{G}^*$), i.e.

$$x + 0y - (\hat{g} + u)z = 0 \ \leftrightarrow \ (-\hat{g} - u, 0, 1) \in \mathcal{G}\overline{a}_0 + \overline{b}.$$

This finishes the proof of Theorem 2, too. ∎

5.2. Proof of Theorem 4

Consider

$$F(x,y) \stackrel{\text{def}}{=} \frac{p(x) - p(y)}{x - y},$$

the slope of the segment which connects two generic points of the curve. After appropriate simplifications, F becomes a polynomial in two variables.

Let $A_i = \big(s_i, p(s_i)\big)$ for $i = 1, \ldots, n$; then F only takes at most Cn distinct values while x and y, independently of each other, range over $\{s_1, \ldots, s_n\}$. Therefore, (see [4], Theorem 2), F can be written in one of the forms

$$(1) \qquad F(x,y) = f(g(x) + h(y)); \text{ or}$$

$$(2) \qquad F(x,y) = f(g(x) \cdot h(y)),$$

for suitable polynomials $f, g, h \in \mathbb{R}[z]$.

Now if $p(x) = a_r x^r + a_{r-1} x^{r-1} + \ldots + a_1 x + a_0$, then the highest (total) degree terms of F are $a_r(x^{r-1} + x^{r-2}y + \ldots + y^{r-1})$. Note that, for $r \geq 3$, there are three or more monomials here and they involve at least one which contains both x and y.

These leading terms must come from the highest power in $f(z)$, with the sum or product of the leading terms of g and h substituted into z, in cases (1) or (2), respectively.

We show that this is impossible.

(1) in case of $F(x,y) = f(g(x) + h(y))$, the coefficients of the highest degree terms on the right hand side come from a binomial expansion and, thus, they cannot be equal — unless $\deg f = 1$. However, in the latter case, no term can involve both x and y simultaneously, a contradiction;

(2) in case of $F(x,y) = f(g(x) \cdot h(y))$, there is just one highest degree term on the right hand side; again a contradiction. ∎

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